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A symmetry of sphere map implies its chaos*

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Abstract. A well-known example, given by Shub, shows that for any $|d| \geq 2$ there is a self-map of the sphere S^n, $n \geq 2$, of degree d for which the set of non-wandering points consists of two points. It is natural to ask which additional assumptions guarantee an infinite number of periodic points of such a map. In this paper we show that if a continuous map $f : S^n \to S^n$ commutes with a free homeomorphism $g : S^n \to S^n$ of a finite order, then f has infinitely many minimal periods, and consequently infinitely many periodic points. In other words the assumption of the symmetry of f originates a kind of chaos. We also give an estimate of the number of periodic points.

Keywords: periodic point, minimal period, homotopy minimal period, equivariant map, Nielsen number.

Mathematical subject classification: Primary: 55M20, 57Bxx; Secondary: 37C80, 5SF20, 54H25.

1 Main results

In discrete dynamical systems theory one of the most natural problems is to study periodic points and minimal periods of a continuous map f. We suppose that $f : X \to X$ is a self-map of a smooth compact manifold X. We shall use the following notation:

$$\begin{aligned} &P^k(f) = \text{Fix}\,(f^k), \quad P_k(f) = \{x \in X : k \text{ is the minimal period of } x\}, \\ &\text{Per}(f) = \{k : P_k(f) \neq \emptyset\}, \quad P(f) = \bigcup_{k\in\mathbb{N}} P^k(f) = \bigcup_{k\in\mathbb{N}} P_k(f). \end{aligned} \tag{1}$$

In the study of periodic points it is important to have a description of the set $\text{Per}(f)$ and a function (sequence) $k \mapsto \#P_k(f)$, or $k \mapsto \#P^k(f)$, where $\#A$

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denotes the cardinality of the set $A \subset X$. In an naive approach to the notion of chaos, one can use the following definition.

Let $f : X \to X$ be a map. We say that f has *chaotical behavior*, or shortly that it originates *chaos*, if either $\mathrm{Per}(f) \subset \mathbb{N}$ is an infinite set or, putting a stronger requirement, if the function $k \mapsto \#P_k(f)$ is unbounded.

It is obvious that the chaotical behavior of f in each of the above senses implies the existence of infinitely many periodic points of f. Studying the latter property of f, Shub and Sullivan showed that if for a map $f : M \to M$ of a compact smooth manifold M the sequence of Lefschetz numbers $\{L(f^m)\}$ of iterations of f is unbounded and f is of class C^1, then it has infinitely many periodic points ([20]).

One can ask whether the statement of the Shub-Sullivan theorem still holds if we drop out the assumption about the smoothness of f. The answer is negative in general, as follows from an example given by Shub [19].

Example 1.1. *Let $h_d : S^1 \to S^1$ be a map of the circle of degree d, e.g. $h_d(z) := z^d$. Further let $\eta : [0,1] \to [0,1]$ be the map given as $\eta(t) = \sqrt{t}$. Representing S^2 as the suspension of S^1, i.e. $S^2 = S^1 \times [0,1]/\sim$ where we contract $S^1 \times \{0\}$ and $S^1 \times \{1\}$ to points. We define a continuous map*

$$f([(z,t]) = [(h_d(z), \eta(t)]) .$$

Then $\deg(f) = \deg(h_d) = d$. It is easy to check that the set of non-wandering points of f (thus also periodic points) consists of two (fixed) points $[S^1 \times \{0\}]$ and $[S^1 \times \{1\}]$, which means that there is not a chaos then. On the other hand $L(f^m) = 1 - d^m$ is unbounded there. Note that f is not differentiable at $[S^1 \times \{0\}]$.

The analogous construction works for a sphere of any dimension $n \geq 2$.

On the other hand there are compact manifolds such that for any continuous self-map of such a manifold the unboundedness of the sequence $\{L(f^m)\}$ implies the existence of infinitely many periodic points. In [2], Block et al., making an attempt to show a Šarkovsky type theorem (cf. [18]) for maps of the circle, proved the theorem stated below. To formulate it we remind that $\mathrm{HPer}(f) \subset \mathrm{Per}(f)$ denotes the set of all minimal periods of f which are minimal periods for every map h homotopic to f (cf. [9, 10, 11]), called the *homotopy minimal periods*.

Theorem 1.2. *Let $f : S^1 \to S^l$ be a map of the circle of degree $\deg(f) = d$. Then*

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $d = 1$.

(F) $\mathrm{HPer}(f)$ *is nonempty and finite if and only if* $d = -1$ *or* $d = 0$. *We have then* $\mathrm{HPer}(f) = \{1\}$.

(G) $\mathrm{HPer}(f)$ *is equal to* $\mathbb{N}$ *for the remaining values of* d, *i.e.* $|d| > 1$, *except is one special case, namely* $d = -2$, *when* $\mathrm{HPer}(f) = \mathbb{N} \setminus 2$. □

In particular if $|\deg(f)| > 1$, then f originates chaos, by Theorem 1.2 (G).

Next a complete description of the set of homotopy minimal periods was given for maps of the two-torus [1], any torus [15], a compact nilmanifold [9], a completely solvable solvmanifold, and a special NR-solvmanifold [11] consecutively. The answer is formulated in a more complicated way than Theorem 1.2. Roughly speaking in the case which is equivalent to the condition that $\{L(f^m)\}$ is unbounded, the set of homotopy minimal periods, thus minimal periods, is infinite as it is in the case for the previously described circle case. An approach is based on Nielsen theory of periodic points [6, 9] due to the geometric properties of the mentioned classes of manifolds. As an application of the approach one can derive Šarkovsky type theorems for mappings of three dimensional nilmanifolds and completely solvable solvmanifolds [10, 11]. On the other hand, the Nielsen theory is useless in studying maps of spheres because S^n is simply-connected if $n \geq 2$. A special position of the circle in this approach is the fact that it is simultaneously a sphere and a torus.

One can ask whether the assumption on the smoothness of f can be replaced by another geometric condition on f to get the statement of the Shub-Sullivan theorem. In this work we show that a continuous map $f : S^n \to S^n$ of degree d, $|d| \geq 2$, gives rise to chaos if it commutes with a free homeomorphism $g : S^n \to S^n$ of finite order larger than 1. More precisely, we prove that $\#\mathrm{Fix}\,(f^k)$ is unbounded as a function of k and the set $\mathrm{Per}(f)$ is infinite (Theorems 1.6, 1.9). Since we will use some facts on transformation group theory, it is convenient to put our symmetry assumption also in the terms of transformation groups.

Definition 1.3. *Let* X *be a smooth manifold and* $g : X \to X$ *be a homeomorphism of finite order* m. *We say that* g *is free if for every* $x \in X$ *and* $1 \leq k \leq m$, $g^k(x) = x$ *implies* $k = m$. *Equivalently, for a homeomorphism* $g : X \to X$ *of order* m *we say that an action of the cyclic group* $\{g\} \equiv \mathbb{Z}_m$ *on* X *is given then by* $(k, x) \mapsto g^k(x)$. *If* g *is free, then this action is called a free action (cf. [3]).*

Definition 1.4. *Let* X *be a smooth manifold with an action of a cyclic group* $\mathbb{Z}_m$ *defined by a homeomorphism* $g\colon X \to X$ *of order* m. *We say that a map* $f : X \to X$ *is* $\mathbb{Z}_m$*-equivariant if* $f\alpha = \alpha f$, *or equivalently, if for the each* $\alpha \in \mathbb{Z}_m$ *and every* $x \in X$, $f\alpha(x) = \alpha f(x)$. *Note that* f *is* $\mathbb{Z}_m$*-equivariant if* *it commutes with the generator of action i.e.* $f(\alpha x) = \alpha f(x)$. *We say that a homotopy* $H : X \times [0,1] \to X$ *is equivariant if*

$$x \in X, \quad t \in [0,1], \quad \alpha \in \mathbb{Z}_m \qquad \textit{imply} \qquad H(\alpha x, t) = \alpha H(x,t)\,.$$

Suppose that we are given a free action of the finite cyclic group $\mathbb{Z}_m$ on the sphere S^n, $n > 2$, i.e. we are given a free homeomorphism $g : S^n \to S^n$ of order m.

Definition 1.5. *Let* $m = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, $\alpha_i > 0$, *be the decomposition of* m *into prime powers. Let next* k *be a natural number. We represent* k *by* $k = p_1^{b_1} \cdots p_s^{b_s}\, p_{s+i}^{a_{s+1}} \cdots p_r^{a_r}$ *where* $p_{s+1}, \dots, p_r$ *are other different primes and* $b_i \geq 0$, $a_i > 0$. *We put*

$$k' := p_1^{b_1} \cdots p_s^{b_s}\,.$$

Now we are in position to formulate our main result.

Theorem 1.6. *Let* $g\colon S^n \to S^n$, $n \geq 1$, *be a free homeomorphism of finite order* $m > 1$, *and* $f : S^n \to S^n$ *be a map of sphere that commutes with* g. *Suppose that* $\deg(f) \notin \{-1,\ 0,\ 1\}$. *Then for every* $k \in \mathbb{N}$ *we have*

$$\#\mathrm{Fix}\,(f^{km}) \;\geq\; m^2\, k'$$

where k' is as in Definition 1.5. In particular, for $k = m^s$ *we have*

$$\#\mathrm{Fix}\,(f^{m^{s+1}}) \;\geq m^{s+2}\,.$$

Corollary 1.7. *Under the above assumptions*

$$\limsup_{l\to\infty} \frac{\#\mathrm{Fix}\,(f^l)}{l} \;\geq m\,. \qquad \square$$

Furthermore, note that for a self-map f of the sphere S^n, $n \geq 1$, the sequence $\{L(f^k)\}$ of the Lefschetz numbers of iterations is unbounded if and only if $\deg(f) \neq 0, \pm 1$ (see Remark 2.3). From Remark 2.3, it follows that Theorem 1.6 replaces the smoothness assumption in the classical Shub-Sullivan theorem [20] by a symmetry assumption in the case of the sphere map.

Corollary 1.8. *Let* $f\colon S^n \to S^n$ *be a continuous map such that the sequence* $\{L(f^n)\}$ *is unbounded. If* f *commutes with a free homeomorphism* $g : S^n \to S^n$ *of order* $m > 1$, *then the set* $P(f)$ *of periodic points is infinite.* $\square$

Let us fix a prime number $p \mid m$ and restrict the action to $\mathbb{Z}_p \subset \mathbb{Z}_m$. In this case we can estimate not only $\# P^l(f) = \#\mathrm{Fix}\,(f^l)$, but also $\# P_l(f)$.

Theorem 1.9. *Let $f\colon S^n \to S^n$ be a continuous map which commutes with a free homeomorphism g of S^n of prime order p. If $\deg(f) \neq \pm 1$, then for each $l \in \mathbb{N}$ there exist at least $p-1$ mutually disjoint orbits of periodic points each of length p^l. Thus*

$$\# P_{p^l}(f) \;\geq (p-1)\, p^l\,.$$

The general idea of the proofs of Theorems 1.6 and 1.9 is to study a map $\bar{f}\colon M \to M$ of the quotient space $M := S^n/\mathbb{Z}_m$ induced by the $\mathbb{Z}_m$-equivariant map $f\colon S^n \to S^n$ in the problem. Next we estimate the number of periodic points of $\bar{f}$, and we "lift" them to periodic points of f. To study periodic points of the induced map $\bar{f}$ we use the Nielsen theory adapted to this situation. It is worth pointing out that a direct application of the Nielsen number is inefficient (see remarks in Section 7).

The paper is organized as follows. First in Section 2 we remind some facts on equivariant maps. In Section 3 we give a brief presentation of the Nielscn theory adapted to the discussed problem. Next in Section 4 we discuss periodic points of a map of the quotient space M to get an estimate of the number of periodic points of a map which is induced by an equivariant map of S^n (Theorem 3.1, Corollary 4.5). In Section 5 we derive an effective form (Theorem 5.6) of the latter formula using a geometric observation (Lemmas 5.1, 5.2) and elementary arithimetical computation (Theorem 5.7). Section 6 contains the proofs of the main theorems 1.6, 1.9.

2 Equivariant maps

In this section we include some facts about equivariant maps which we will need.

Proposition 2.1. *Suppose that $\mathbb{Z}_m$ acts freely on S^n, $n \geq 1$. If $f\colon S^n \to S^n$ is an equivariant map, then*

$$\deg(f) \equiv 1 \mod m\,. \qquad \square$$

The above fact is well known and has various proofs. We remark only that for $m = 2$, this is the classical Borsuk-Ulam theorem which states that an odd map has odd degree.

Recall that the degree of a map classifies homotopy classes of (non-equivariant) maps of the sphere S^n.

The following theorem was proved by R. Rubinsztein [17].

Theorem 2.2. *Suppose that a finite group G acts freely on S^n, $n > 1$.*

Then the natural function $[S^n, S^n]_G \to [S^n, S^n]$ of the set of equivariant homotopy classes into the set of homotopy classes is an injection, i.e. if two equivariant maps have the same degree, then they are equivariantly homotopic. Moreover the image of $[S^n, S^n]_G$ in $[S^n, S^n] = \mathbb{Z}$ is equal to $\{m\mathbb{Z} + 1\}$.

Remark 2.3. Observe that from Theorem 2.1 it follows that $\deg(f) = lm + 1$ for any equivariant map. Consequently the assumption $\deg f \neq \pm 1$ means then $|\deg f| > 1$ if $m > 2$. If $m = 2$ there are equivariant maps of degree -1, but $\deg f = 0$ is excluded in this case.

For a better ilustration of the idea of the conclusion of Theorems 1.6 and 1.9 we present the following example about the dynamics of the canonical equivariant maps of the unit circle with a free action of the group $\mathbb{Z}_m$ of roots of unity.

Example 2.4. *For a given m, let the generator of cyclic group $\mathbb{Z}_m$ act (freely) on S^1 by rotation by the angle $\frac{2\pi i}{m}$, i.e. the subgroup of roots of unity of degree m acts on the whole group. It is easy to check that $f(x) := z^{lm+1}$, $0 \neq l \in \mathbb{Z}$, is a $\mathbb{Z}_m$-equivariant map of the circle. Note that $f^r(z) = z^{(lm+1)^r}$, and by definition z is an r-periodic point if $z^{(lm+1)^r} = z$ and r is the smallest number with this property. It is equivalent to the fact that z is a root of unity of degree $(lm+1)^r - 1$ but not of degree $(lm+1)^{r'} - 1$ with $r' \mid r$. Let us consider all the iterations as consecutive powers of a natural number $m > 1$, i. e. $r = m^s$. It is easy to check the following. If $a, b \in \mathbb{Z}$, $\alpha \in \mathbb{N}$, and $m \geq 2$, then*

$$a \equiv b \mod m^\alpha \Longrightarrow \quad a^m \equiv b^m \mod m^{\alpha+1}.$$

Applying this s times to $a = lm + 1$ and $b = 1$ we get

$$(lm+1)^{m^s} \equiv 1 \mod m^{s+1}, \quad \text{i.e.} \quad m^{s+1} \mid (lm+1)^{m^{s+1}} - 1.$$

Consequently for any $s > 0$, roots of unity of degree m^{s+1} are roots of the polynomial $z^{(lm+1)^{m^s}} - z$, i.e. they belong to $P^{m^s}(f) = \mathrm{Fix}(f^{m^s})$. This gives the following estimate

$$\#\,\mathrm{Fix}(f^{m^s}) \geq m^{s+1} = mm^s. \tag{2}$$

Among all roots of unity of degree m^{s+1} there are $\phi(m^{s+l})$ primitive roots of degree m^{s+1}, where $\phi(k)$ is the Euler function, i.e. the number of all numbers less then k and relatively prime to k. We show that these roots belong to $P_{m^s}(f)$.

It is enough to show that for a primitive root of unity ξ of degree m^{s+1} we have $f^{m^{s-1}}(\xi) \neq \xi$. Indeed

$$\xi^{(lm+1)^{m^{s-1}}} = (\xi^{lm})^{m^{s-1}} = \xi^{ml^s}\xi^{m^s l(m-1)}\xi^{m^s l(1-m)}\xi^{m^{s-1}} = 1 \cdot \xi^{-(m^{s+1}l - m^s l + m^{s-1})},$$

because ξ is a primitive root of unity of degree m^{s+1} and $m^s l - m^{s-1} \not\equiv 1$ mod m^{s+1}. *Since $\phi(m^{s+1}) = m^s\phi(m^s)$, the above shows that*

$$\#P_{m^s}(f) \;\geq m^s\phi(m) \tag{3}$$

for the above map. In particular if $m = p$ is a prime, then

$$\#P_{p^s}(f) \;\geq\; p^s\phi(p) \;=\; p^s(p-l)\,. \tag{4}$$

Note that taking the suspension of this map we get a map Σf of S^2 with the same dynamics as f of Example 2.4. On the other hand, slightly modifying $\eta(t)$ of Example 1.1 we can construct a map of S^2 which is a small perturbation of Σf but has only two non-wandering points. Theorems 1.6 and 1.9 say that any small equivariant perturbation, or more generally any equivariant continuous deformation of f must possess at least the part of dynamics described above.

3 Nielsen Theory

We recall briefly the facts of Nielsen theory. For the details we refer the reader to [12].

A few words about the notation. Usually the covering maps are denoted by $p\colon \tilde{X} \to X$ and we will do so in this section. However in the rest of the paper we will be given a space X with a free action of a finite group G on X. This yields a covering $X \to \bar{X} = X/G$ onto the orbit space. We will denote this covering $p\colon X \to \bar{X}$.

Let $p\colon \tilde{X} \to X$ be a universal covering of a polyhedron. We denote by

$$\mathcal{O}_X \;:=\; \{\alpha : \tilde{X} \to \tilde{X} :\; p\alpha = p\}$$

the group of deck transformations of this covering. This group has a (non-canonical) bijection with the fundamental group $\pi_1 X$ although we will not use this correspondence in this paper. Let $f\colon X \to X$ be a map and let lift$(f) = \{\tilde{f}\colon \tilde{X} \to \tilde{X}\colon p\tilde{f} = fp\}$ denote the set of all lifts of f. If we fix a lift $\tilde{f}_0$, then each other lift of f can be uniquely written as $\alpha\tilde{f}_0$, $\alpha \in \mathcal{O}_X$. Consider the action of $\mathcal{O}_X$ on the set lift(f) given by

$$\alpha \circ \tilde{f} \;=\; \alpha\tilde{f}\alpha^{-1}\,.$$

The orbits of this action are called *Reidemeister classes* and their set is denoted by $\mathcal{R}(f)$.

On the other hand we consider the fixed point set:

$$\mathrm{Fix}\,(f) := \{x \in X : f(x) = x\}.$$

We define the *Nielsen relation* on this set as follows. We say that two fixed points x , y are *Nielsen related* if there is a path $\omega : [0,1] \to X$ satisfying: $\omega(0) = x$, $\omega(1) = y$ and moreover the paths ω and $f\omega$ are homotopic rel $\{0,1\}$. This relation divides $\mathrm{Fix}\,(f)$ into a finite number of mutually disjoint classes. We denote the set of these classes by $\mathcal{N}(f)$. It turns out that, for any lift $\tilde{f} \in \mathrm{lift}(f)$, the set $p(\mathrm{Fix}\,(\tilde{f}))$ is either a Nielsen class of f or is the empty set. Each Nielsen class is of the above form. Moreover subordinating to a Nielsen class $A \subset \mathrm{Fix}\,(f))$ a lift $\tilde{f} \in \mathrm{lift}(f)$ satisfying $A = p(\mathrm{Fix}\,(\tilde{f}))$ we get the map $j : \mathcal{N}(f) \to \mathcal{R}(f)$ which is injective (but is not onto in general). Thus we may identify each Nielsen class with a Reidemeister class. On the other hand the restriction of f to $\mathrm{Fix}\,(f^k)$ is a natural homeomorphism which induces the self-map of $\mathcal{N}(f^k)$ and the last extends to the self-map $\mathcal{R}_f : \mathcal{R}(f^k) \to \mathcal{R}(f^k)$ given by $\mathcal{R}_f[h] = [h']$, where $h' \in \mathrm{lift}(f^k)$ is the unique lift making the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{h} & \tilde{X} \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ \tilde{X} & \xrightarrow{h'} & \tilde{X} \end{array}$$

commutative (for a fixed lift $\tilde{f}$ of f). Since $(\mathcal{R}_f)^k = \mathrm{id}$, we get an action on the group $\mathbb{Z}_k$ on $\mathcal{R}(f^k)$. The orbits of this action are called *orbits of Reidemeister classes* and their set is denoted by $\mathcal{OR}(f^k)$. Now we consider the natural map

$$\mathrm{lift}(f) \ni \tilde{f} \mapsto \tilde{f}^k \in \mathrm{lift}(f^k).$$

This induces the map $i_{k1} : \mathcal{R}(f) \to \mathcal{R}(f^k)$. Similarly we define $i_{kl} : \mathcal{R}(f^l) \to \mathcal{R}(f^k)$ for $l \mid k$. A Reidemeister class $A \in \mathcal{R}(f^k)$ class is called *reducible* if $A = i_{kl}(B)$ for $B \in \mathcal{R}(f^l)$, for an $l \mid k$, $l < k$. An orbit of Reidemeister classes is called *reducible* if one (hence all) of its elements is a reducible Reidemeister class. In [12] Boju Jiang introduced a number $NF_k(f)$ which is ahomotopy invariant and is the lower bound for the cardinality of $\mathrm{Fix}\,(f^k)$ (of the self map $f : X \to X$ of a finite polyhedron). Here we do not need to recall (a little complicated) definition of $NF_k(f)$, since in the case when all involved Reidemeister classes are essential this invariant is equal to the sum given in the next Theorem (see Chapter 3 of [12]).

Theorem 3.1. *For any self-map $f : X \to X$ of a finite polyhedron and a fixed natural number $k \in \mathbb{N}$*

$$\#Fix(f^k) \geq \sum_{r|k} (\#\mathcal{IEOR}(f^r)) \cdot r$$

where $\mathcal{IEOR}(f^r)$ denotes the set of irreducible ($\mathcal{I}$) essential ($\mathcal{E}$) orbits ($\mathcal{O}$) of Reidemeister ($\mathcal{R}$) classes of the map f^r.

Proof. The inequality follows from:

1. each essential Reidemeister class (considered as the Nielsen class) is non-empty,
2. irreducible Reidemeister classes are mutually disjoint,
3. each irreducible essential orbit of Reidemeister classes in $\mathcal{IEOR}(f^r)$ contains at least r periodic points (of period r). □

4 Periodic points of a self-map of the quotient space

In this section $M = S^n/\mathbb{Z}_m$ $(m > 1)$ will denote the quotient space of a free action, as above, and $\bar{f} : M \to M$ will denote the self map induced by an equivariant map $f : S^n \to S^n$ of degree $\neq 0, \pm 1$. With respect to Proposition 2.1 it is enough to assume that $\deg(f) \neq \pm 1$, or only $\deg(f) \neq 1$ if $m \geq 3$. We will give an estimate for the number of periodic points of the equivariant map f. Since $p(\mathrm{Fix}\,(f^k)) \subset \mathrm{Fix}\,(\bar{f}^k)$, we first consider the periodic points of the map $\bar{f}$. We will use the formula from Theorem 3.1. We will show that under our assumptions, all involved Reidemeister classes of $\bar{f}$ and of its iterations are essential and each orbit of Reidemeister classes consists of one element.

Lemma 4.1. *Consider the commutative diagram*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & Y \end{array}$$

where $p : \tilde{Y} \to Y$ is a finite regular covering of a finite polyhedron Y. Then

$$\mathrm{ind}\,(\tilde{f}) = r \cdot \mathrm{ind}\,(f;\, p(\mathrm{Fix}\,(\tilde{f})))$$

where $r = \#\{\alpha \in \mathcal{O}_Y;\ \tilde{f}\alpha = \alpha\tilde{f}\}$ *(*$\mathcal{O}_Y$ *denotes the group of covering transformations of the regular covering* p*; exceptionally in this Lemma we do not need to assume that the covering* p *is universal). In particular* $\mathrm{ind}\ (f;\ p(\mathrm{Fix}\ (\tilde{f}))) \neq 0$ *if and only if* $L(\tilde{f}) = \mathrm{ind}\ (\tilde{f}) \neq 0$.

Proof. If $\mathrm{Fi\tilde{x}}\ (f) = \emptyset$, then both sides are zero. Suppose that there is a point $\tilde{x} \in Fix\,(\tilde{f})$ and let $\alpha \in \mathcal{O}_Y$. Then

$$\alpha\tilde{x} \in \mathrm{Fix}\,(\tilde{f}) \iff \tilde{f}(\alpha\tilde{x}) = \alpha\tilde{x} \iff \tilde{f}\alpha(\tilde{x}) = \alpha\tilde{f}(\tilde{x}) \iff \tilde{f}\alpha = \alpha\tilde{f}\,.$$

Thus $\# p^{-1}(x) \cap \mathrm{Fix}\ \tilde{f} = r$. Since both sides of the equality are homotopy invariant, we may assume that Fix (f) is finite. Since the covering map is a local homeomorphism,

$$\mathrm{ind}\ (f) = \sum_x \mathrm{ind}\ (\tilde{f};\ p^{-1}(x)) = \sum_x r \cdot \mathrm{ind}\,(f;x) = r \cdot \mathrm{ind}\,(f;\ p(\mathrm{Fix}\ (\tilde{f})))$$

where x runs through the set $p(\mathrm{Fix}\ (\tilde{f}))$. □

Corollary 4.2. *Let* $\bar{f} : M \to M$ *be the map induced by an equivariant map* $f : S^n \to S^n$ *of degree* $\neq 0, \pm 1$. *Then all the Reidemeister classes of* f *and of all its iterations are essential.*

Proof. The assumption that $\deg(f) \neq 0, \pm 1$ implies the same inequality for all other lifts of $\bar{f}$ (and their iterations). Thus $L(\bar{f}^k) \neq 0$, which implies, by Lemma 4.1, that all the Reidemeister classes of $\bar{f}^k$ are essential. □

Lemma 4.3. *If a self-map of the orbit space* $\bar{X} = X/G$*, of a free action of a finite group* G*, is induced by an equivariant map* $f : X \to X$ *then the map* $\mathcal{R}_{\bar{f}} : \mathcal{R}(\bar{f}^k) \to \mathcal{R}(\bar{f}^k)$ *is the identity. Thus each orbit of Reidemeister classes consists of exactly one element.*

Proof. Let us recall that each lift of $\bar{f}^k$ is of the form αf^k where $\alpha \in \mathcal{O}_X$. Since f commutes with α as an equivariant map, $f(\alpha f^k) = (\alpha f^k) f$. Moreover $\mathcal{R}_{\bar{f}}[h] = [h']$ if the lifts $h,\ h' \in lift(\bar{f})$ satisfy $fh = h'f$. Thus for $h = \alpha f^k$ we may put $h = h'$, and hence $\mathcal{R}_{\bar{f}}[h] = [h]$ for any $[h] \in lift(\bar{f})$. □

Lemma 4.4. *The Reidemeister relation of the map* $\bar{f} : \bar{X} \to \bar{X}$ *induced by an equivariant map* $f : X \to X$ *is trivial. Thus* $\mathcal{R}(\bar{f}) = \mathcal{O}_{\bar{X}} = \mathbb{Z}_m$.

Proof. Each element of lift($\bar{f}$) is of the form αf ($\alpha \in \mathcal{O}_X$). The Reidemeister action is given by $\beta \circ (\alpha f) = \beta\alpha f \beta^{-1}$. Since f is equivariant, it commutes with the maps $\alpha,\ \beta : X \to X$. This and the commutativity of $\mathcal{O}_X = \mathbb{Z}_m$ imply

$$\beta \circ (\alpha f) \ = \ \beta\alpha f \beta^{-1} = \alpha f \ . \qquad \square$$

Corollary 4.5. *If $\bar{f} : M \to M$ ($M = S^n/\mathbb{Z}_m$) is a map induced by an equivariant map $f : S^n \to S^n$, then we have*

$$\#\mathrm{Fix}\ (\bar{f}^k) \ \geq \ \sum_{r|k} (\#\mathcal{IR}(\bar{f}^r)) \cdot r$$

Proof. The equality follows from Theorem 3.1 once we notice that in each summand on the right hand side $\mathcal{IEOR} = \mathcal{IR}$. In fact Lemma 4.2 allows to drop $\mathcal{E}$ and Lemma 4.3 allows to drop the symbol $\mathcal{O}$. $\square$

Thus it remains to find the number of irreducible classes in $\mathcal{R}(\bar{f}^r)$. Let us recall that the class $A \in \mathcal{R}(\bar{f}^k)$ is reducible iff it belongs to the image of the map $i_{kl} : \mathcal{R}(\bar{f}^l) \to \mathcal{R}(\bar{f}^k)$ for an $l \mid k$, $l < k$.

5 The lower bound of the number of periodic points

In this section we will give formula for the right hand side of the inequality in Corollary 4.5. Recall that by Lemma 4.4 we may identify $\mathcal{R}(\bar{f}) = \mathbb{Z}_m$. Moreover the map $i_{kl} : \mathbb{Z}_m \to \mathbb{Z}_m$ is given by $i_{kl}(s) = rs$ where $r = k/l$. To prove the last we recall that in general $i_{kl}[a] = [a^{k/l}]$. Since the isomorphism $\mathbb{Z}_m \equiv \mathcal{R}(\bar{f}^k)$ is given by $s \leftrightarrow a^s$ (where a is a fixed generator of $\pi_1(M)$), $i_{kl}[a] = [a^{k/l}]$ corresponds to $i_{kl}(s) = k/l \cdot s$.

We say that a natural number r *eventually divides* m if r divides a power m^s. In other words r eventually divides m if and only if for a prime number p

$$p|r \Rightarrow p|m$$

.

Let us notice that then the number k' defined in Definition 1.5 equals the greatest divisor of k that eventually divides m.

We consider two cases.

(**i**) r does not eventually divide m

Lemma 5.1. *Suppose that r does not eventually divide m. Then*

$$\#\mathcal{IR}(\bar{f}^r) \ = \ 0 \, .$$

Proof. Let p be a prime number which divides r but does not divide m. Then the map

$$i_{r,r/p}\colon \mathcal{R}(\bar{f}^{r/p}) = \mathbb{Z}_m \to \mathcal{R}(\bar{f}^r) = \mathbb{Z}_m$$

is given by $i_{r,r/p}[a] = [pa]$. Since p and m are relatively prime, the map $i_{r,r/p}$ is onto which makes each class in $\mathcal{R}(\bar{f}^r)$ reducible. □

(ii) Now we assume that r eventually divides m.

We have the following

Lemma 5.2. *Let r eventually divide m. Then the class $a \in \mathbb{Z}_m = \mathcal{R}(\bar{f}^r)$ is reducible iff the numbers a, r are not relatively prime.*

Proof. $\Longleftarrow$ Let $d := \gcd(a, r) > 1$. Then $i_{r,r/d}$ is sending

$$\mathcal{R}(\bar{f}^{r/d}) = \mathbb{Z}_m \ni a/d \mapsto a \in \mathbb{Z}_m = \mathcal{R}(\bar{f}^r)\,,$$

hence the class $a \in \mathbb{Z}_m$ is reducible.

$\Longrightarrow$ Let $a = i_{rl}(b)$ where $b \in \mathbb{Z}_m = \mathcal{R}(\bar{f}^l)$, $l < k$, $l \mid k$. Then $r/l \cdot b \equiv a \,(mod\, m)$. Let p be a prime dividing the number $r/l > 1$. Then $p \mid l$ implies $p \mid m$ and by the above congruence we get $p \mid a$. Thus $\gcd(a, m) \geq p > 1$. □

To formulate and to study the number of Reidemeister classes (Nielsen classes) of mappings of M it is useful to introduce the following arithmetic function. It also seems be interesting by itself.

Definition 5.3. *For a given $m \in \mathbb{N}$ we define a function $\phi_m : \mathbb{N} \to \mathbb{N}$ by*

$$\phi_m(k) \;:=\; \#\{a \in \mathbb{N} : a \text{ and } k \text{ are relatively prime, and } a \leq m\}\,.$$

Remark 5.4. Notice that for $k = m > 1$

$$\phi_m(m) \;=\; \text{the cardinality of the set of natural numbers } < m \text{ relatively prime with } m$$

equals the Euler function. However for $k = 1$ we have $\phi_m(1) = m$ while the Euler function $\phi(1) = 0$.

As a consequence of Lemma 5.2 we get.

Corollary 5.5. *For a map $\bar{f}\colon M \to M$ induced by a $\mathbb{Z}_m$-equivariant map $f\colon S^n \to S^n$ and for r eventually dividing m we have*

$$\mathcal{IEOR}(\bar{f}^r) \;= \mathcal{IR}(\bar{f}^r) = \phi_m(r)\,,$$

where $\phi_m(k)$ is defined above. □

Theorem 5.6. *For a map* $\bar{f}\colon M \to M$ *induced by a* $\mathbb{Z}_m$*-equivariant map* $f : S^n \to S^n$*, we have*

$$\#\mathrm{Fix}\ (\bar{f}^k) \ \geq \ \sum_{l|k} \phi_m(l) \cdot l\,,$$

where the sum is taken over all divisors l *of* k *eventually dividing* m*.*

Proof. Note that by Lemma 5.1, in the sum of Corollary 4.5 we may omit out the summands in which r which does not eventually divide m, as follows from Lemma 5.1. Now the statement follows from Corollary 5.5. □

Now we prove the main arithmetic formula deriving the right hand side of Theorem 5.6.

Theorem 5.7. *For a fixed* $m \in \mathbb{N}$ *and any* $k \in \mathbb{N}$ *we have*

$$\sum_{l|k} \phi_m(l) \ = \ m \cdot k'\,,$$

where the sum is taken over all divisors l *of* k*, that eventually divide* m *and* k' *is given by Definition 1.5.*

Proof. Let us recall that a divisor $l \mid k$ eventually divides m iff it is a divisor of k'. Consequently the equality of the statement reduces to $\sum_{l|k'} \phi_m(l) \ = \ m \cdot k'$, where the sum is taken over all divisors of k'. Equivalently it is enough to show that for a natural number k eventually dividing m we have

$$\sum_{l|k} \phi_m(l) \ = \ m \cdot k$$

where the sum is taken over all divisors of k.

Then we may represent $m = p_1^{a_1} \cdots p_s^{a_s}$, where $a_1, \ldots, a_s \geq 1$ and $k = p_{i_1}^{b_1} \cdots p_{i_t}^{b_t}$, where $t \leq s$ and $b_1, \ldots, b_t \geq 1$. The sum from the Theorem splits:

$$\sum_{l|k} \phi_m(l) \cdot l \ = \ \sum_0 + \sum_1 + \cdots + \sum_t\,,$$

where $\sum_\gamma$ is taken over the numbers l divisible by exactly γ distinct primes. Then

$$\sum_\gamma \ = \ \sum_{i_1,\ldots,i_\gamma} \ \sum_{j_1,\ldots j_\gamma} \phi_m(p_{i_1}^{j_1} \cdots p_{i_\gamma}^{j_\gamma}) \cdot p_{i_1}^{j_1} \cdots p_{i_\gamma}^{j_\gamma}$$

where $1 \leq i_1 < \ldots i_\omega \leq i_s \leq t$ and $1 \leq j_1 \leq b_{i_1}, \ldots, 1 \leq j_\omega \leq b_{i_\omega}$.

By Lemma 5.8 the above sum is equal to

$$= \sum_{i_1,\ldots,i_\gamma} \sum_{j_1,\ldots,j_\gamma} m \cdot \frac{1}{p_{i_1}}(p_{i_1}-1) \cdots \frac{1}{p_{i_\gamma}}(p_{i_\gamma}-1) p_{i_1}^{j_1} \cdots p_{i_\gamma}^{j_\gamma}$$

$$= \sum_{i_1,\ldots,i_\gamma} \left(m \cdot \frac{1}{p_{i_1}}(p_{i_1}-1) \cdots \frac{1}{p_{i_\gamma}}(p_{i_\gamma}-1) \left(\sum_{j_1=1}^{b_{i_1}} p_{i_1}^{j_1} \right) \cdots \left(\sum_{j_\gamma=1}^{b_{i_\gamma}} p_{i_\gamma}^{j_\gamma} \right) \right)$$

$$= \sum_{i_1,\ldots,i_\gamma} m \cdot \frac{1}{p_{i_1}}(p_{i_1}-1) \cdots \frac{1}{p_{i_\gamma}}(p_{i_\gamma}-1) \frac{p_{i_1}^{b_{i_1}+1} - p_{i_1}}{p_{i_1}-1} \cdots \frac{p_{i_\gamma}^{b_{i_\gamma}+1} - p_{i_\gamma}}{p_{i_\gamma}-1}$$

$$= \sum_{i_1,\ldots,i_\gamma} m \cdot (p_{i_1}^{b_{i_1}} - 1) \cdots (p_{i_\gamma}^{b_{i_\gamma}} - 1)$$

Now

$$\sum_0 + \sum_1 + \cdots + \sum_t = \sum_{\gamma=0}^{t} \left(\sum_{i_1,\ldots,i_\gamma} m \cdot (p_{i_1}^{b_{i_1}} - 1) \cdots (p_{i_\gamma}^{b_{i_\gamma}} - 1) \right)$$

$$= m(1 + (p_1^{b_1} - 1)) \cdots (1 + (p_k^{b_k} - 1)) = m(p_1^{b_1} \cdots p_k^{b_k}) = m \cdot k\,.$$

This proves the statement. □

Lemma 5.8. *Let $p_1, \ldots, p_\omega$ be different prime numbers, that divide $m \in \mathbb{N}$. Then*

1. $\phi_m(p_1^{h_1} \cdots p_\omega^{h_\omega}) = \phi_m(p_1 \cdots p_\omega)$ *(for all $h_1, \ldots, h_\omega \in \mathbb{N}$),*
2. $\phi_m(p_1 \cdots p_\omega) = m(1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_\omega})$.

Proof. Ad 1. We notice that for any natural number r:

$$\gcd(p_1^{h_1} \cdots p_\omega^{h_\omega}, r) = 1 \iff \gcd(p_i, r) = 1$$

$$\text{for every } i = 1, \ldots, k \iff \gcd(p_1 \cdots p_\omega, r) = 1\,.$$

Ad 2. Let us denote $A_i = \{h \in \mathbb{N} : h \leq n,\ p_i \mid h\}$. Then $\phi_m(p_1 \cdots p_\omega) = m - \#\bigcup_{i=1}^{\omega} A_i$. Let us notice that (for $1 \leq i_1 < \ldots < i_s \leq \omega$)

$$\#A_{i_1} \cap \ldots \cap A_{i_s} = \frac{m}{p_{i_1} \cdots p_{i_s}}.$$

Now by the inclusion-exclusion principle we have

$$\begin{aligned}\#\bigcup_{i=1}^{\omega} A_i &= -\sum_{s=1}^{\omega}(-1)^s \left(\sum_{1\le i_1<\cdots<i_s\le\omega} \#A_{i_1}\cap\cdots\cap A_{i_s}\right)\\ &= -\sum_{s=1}^{\omega}(-1)^s \left(\sum_{1\le i_1<\cdots<i_s\le\omega} \frac{m}{p_{i_1}\,\cdots\,p_{i_s}}\right).\end{aligned}$$

Thus

$$\phi_m(p_1\cdots p_\omega) = m - \#\bigcup_{i=1}^{\omega} A_i = m - \left[-\sum_{s=1}^{\omega}(-1)^s \sum_{1\le i_1<\cdots<i_s\le\omega} \frac{m}{p_{i_1}\cdots p_{i_s}}\right]$$

$$= m\left[1+\sum_{s=1}^{\omega}(-1)^s \sum_{1\le i_1<\cdots<i_s\le\omega} \frac{1}{p_{i_1}\cdots p_{i_s}}\right] = m(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_\omega}).$$

□

Remark 5.9. Note that for $m = p_1^{a_1}\,\cdots\,p_s^{a_s}$ by Lemma 5.8

$$\phi_m(m^s) = \phi_m(p_1\,\cdots\,p_s) = m(1-\frac{1}{p_1})\,\cdots\,(1-\frac{1}{p_s}) = \phi_m(m) = \phi(m)\,,$$

and consequently the last term of the sum of Theorem 5.6 which corresponds to m^s-periodic points (cf. Example 2.4) is of the form

$$\phi_m(m^s) = m^s\phi(m)\,.$$

Our result can be stated in the following combinatorial way. It can be used for a construction of an algorithm for estimating the cardinality of periodic points of a map as in Theorem 5.6.

Proposition 5.10. *Let $\bar{f}: M \to M$ be as in Theorem 5.6 and k a natural number. If for each prime $p \mid k \Rightarrow p \mid m$, then the number of periodic points of the map $\bar{f}^k$ whose minimal periods are of the form $p_{i_1}^{j_1}\,\cdots\,p_{i_\gamma}^{j_\gamma}$, where $1 \le j_1 \le a_{i_1},\,\ldots\,, 1 \le j_\gamma \le a_{i_\gamma}$, is not less than the coefficient at $\frac{x_1\ldots x_m}{x_{i_1}\ldots x_{i_\gamma}}$ of the polynomial*

$$W(x_1,\,\ldots\,,x_k) = (x_1 + (p_1^{b_1}-1))\,\cdots\,(x_k + (p_k^{b_k}-1))\,.$$

In the general case the same inequality holds but the polynomial W is derived for the numbers k', m.

6 Proofs of main theorems

Now we come back to the equivariant map $f : S^n \to S^n$. This map is the fixed lift of the induced map $\bar{f} : M \to M$ to the universal covering. Thus all periodic points of f are contained in the fibres over periodic points of $\bar{f}$.

Proof of Theorem 1.6. From Theorems 5.6, 5.7 we get

$$\#\mathrm{Fix}\,(\bar{f}^k) \;\geq\; \sum_{l|k} \phi_m(l) \cdot l \;=\; mk' .$$

The lemma below gives m fixed points of f^{km} over each fixed point of $\bar{f}^k$. Thus

$$\#\mathrm{Fix}\,(f^{km}) \;\geq m \cdot \#\mathrm{Fix}\,(\bar{f}^k) \;\geq\; m(m \cdot k') \;=\; m^2 k' . \qquad \square$$

Lemma 6.1. *Let $f : S^n \to S^n$ be a $\mathbb{Z}_m$-equivariant map and $\bar{f} : M \to M$ the map induced by f on the quotient space.*

If $\bar{x} \in \mathrm{Fix}\ \bar{f}^k$, then $p^{-1}(\bar{x}) \subset \mathrm{Fix}\,(f^{mk})$.

Consequently if $\#\ \mathrm{Fix}\,(\bar{f}^k) \geq c(\bar{f}, k)$, then $\#\mathrm{Fix}\,(f^{km}) \geq m\, c(\bar{f}, k)$.

Proof. To shorten notation denote $c(\bar{f}, k)$ by c. Suppose that $\bar{x}_1, \ldots, \bar{x}_c$ are distinct fixed points of $\bar{f}^k$. Consider the fibres over the fixed points $\bar{x}_1, \ldots, \bar{x}_c \in \mathrm{Fix}\,(\bar{f})$. Let us fix a point $x_i \in p^{-1}(\bar{x}_i)$. Then $f^k(x_i) = \alpha_i x_i$ for an $\alpha_i \in \mathcal{O}_M = \mathbb{Z}_m$. Note that x_i is not a fixed point of f if $\alpha_i \neq 1$. (We use the multiplicative notation for the operation in the cyclic group $\mathbb{Z}_m$). Now

$$f^{km}(x_i) \;=\; f^{k(m-1)}(\alpha_i x_i) \;=\; \cdots \;=\; \alpha_i^m x_i \;=\; x_i ,$$

because $\alpha^m = 1$ for every element α of a group of order m. Thus all m elements of the fibre $p^{-1}(\bar{x}_i)$ are fixed points of f^{km}. $\square$

Proof of Theorem 1.9. Let $m = p$ be a prime. Since (by Lemma 4.4) $\mathcal{R}(\bar{f}^{p^k}) = \mathcal{O}_M = \mathbb{Z}_p$, each Reidemeister class consists of a single lift $\alpha^i f^{p^k}$, $1 \leq i \leq p$, where $\alpha \in \mathcal{O}_M$ is a fixed generator. Moreover f^{p^k} is the reducible class (it reduces to $f \in \mathrm{lift}(\bar{f})$) while all remaining $p-1$ Nielsen classes are irreducible (Corollary 4.3). As we have noticed above, each of these classes is a singleton $\{\alpha^i f^{k^k}\}$, denoted shortly by $\alpha^i f^{k^k}$, where $i = 1, \ldots, p-1$. Since $\mathrm{ind}\,(f^{p^k}) \neq \pm 1$, $\mathrm{Fix}\,(f^{p^k}) \neq \emptyset$ hence $p(\mathrm{Fix}\,(f^{p^k})) \neq \emptyset$ is a reducible Nielsen class of $\bar{f}^{p^k}$. On the other hand $p(\mathrm{Fix}\,(\alpha^i f^{p^k}))$, for $i = 1, \ldots, p-1$, are the remaining Nielsen classes. We choose a point $\bar{x}_i \in p(\mathrm{Fix}\,(\alpha^i f^{p^k}))$ for $i = 1, \ldots, p-1$. We will show that all points in the fibre over $\bar{x}_i \in p(\mathrm{Fix}\,(\alpha^i f^{p^k}))$ (for $i = 1, \ldots, p-1$) are periodic points of f with minimal period p^{k+1}. In fact let $x_i \in \mathrm{Fix}\,(\alpha^i f^{p^k})$. Then $x_i = \alpha^i f^{p^k}(x_i)$ and since f is equivariant, the same equality holds for every point of the fibre $p^{-1}(\bar{x}_i)$. Now for each element in this fibre $f^{2p^k}(x_i) = f^{p^k}(\alpha^i x_i) = \alpha^{2i} x_i$ and inductively we get $f^{rp^k}(x_i) = \alpha^{ri} x_i$. Since p is prime, p does not divide ri for $r < p$. Thus $pp^k = p^{k+1}$ is the least period of x_i with respect to f.

It remains to recall that each irreducible essential orbit of $\bar{f}^{p^k}$ has at least p^k elements. Since there are $p-1$ irreducible classes and each fibre contains p elements, we get at least $p^k(p-1)p = (p-1)p^{k+1}$ periodic points of f of the minimal period p^{k+1}. □

7 Final remarks

First we would like to emphasize that the Nielsen theory has been already used to study periodic points in [1], [4], [5], [6], [7, 8, 9, 10, 11], [12, 14, 15]. In all these papers the crucial point is that for the asymptotic Nielsen number

$$N(f^\infty) \;:=\; \limsup \sqrt[k]{N(f^k)} \;>\; 1$$

(cf. [14] for the definition). Let us remark that in our consideration we can not use this argument as follows from the Remark below.

Remark 7.1. For any map $\bar{g}$ of the quotient space $M = S^n/\mathbb{Z}_m$ and every $k \in \mathbb{N}$ we have

$$N(\bar{g}^k) \leq m = \#\pi_1(M)\,,$$

because we have at most m Reidemeister (Nielsen) classes. Consequently $N(g^\infty) = 1$.

Remark 7.2. Secondly, we must also say that our estimate of the number of periodic points of a self map of $M = S^n/\mathbb{Z}_m$ (Cor. 4.5) holds only for a map $\bar{f}$ of M which is induced by an equivariant map f of S^n. Recall that the homotopy invariant $NF_k(g)$, being a lower bound of the cardinality of $\#\mathrm{Fix}\,(g)$, was introduced by Boju Jiang in Chapter 3 of [12]. Recently the first author proved that: in the case of a compact manifold of dimension > 3, $NF_k(g)$ is the best homotopy invariant estimating $\#\mathrm{Fix}\,(g)$ from below i.e. for every g there exists $h : M \to M$ homotopic to g and for which $\#\,\mathrm{Fix}\,(h) = NF_k(g)$ (cf. [8]). In our paper, as well as in all quoted papers [1], [6], [11, 9, 10], [12, 14, 15], [16], this invariant is equal to the sum $\sum_{l|k} NP_l(g)$, where $NP_l(g) = \mathcal{IEOR}(f^r)$ (see

Theorem 3.1) which allows to get the simple formulae. The mentioned equality was possible since all the Reidemeister classes were essential. A similar situation appeared in the papers from the above list, and one would try to repeat the same argument for the map $\bar{f}$ of the orbit space.

On the other hand, in the problems discussed in the papers [1], [6], [11, 9, 10] [12, 14, 15], [16] the fundamental group is infinite, there are infinitely many Nielsen classes, and for any map $NF_k(\bar{f}) = N(\bar{f}^k)$, for every k. Moreover, by the same reason that we work with $NF_k(\bar{f})$ (which is greater than $N(\bar{f}^k)$ here) the information about Nielsen and Reidemeister numbers of all iterations and the Nielsen or zeta function (cf. [4], [5]) of $\bar{f}$ is not considered.

Remark 7.3. It seems be of the interest to study the dynamics of equivariant maps not only for the spheres. In particular we expect that, for any compact closed manifold X with a free action of a finite group G, an analog of Theorem 1.6 holds for an equivariant self-map $f : X \to X$ such that the sequence $\{L(f^k)\}$ is unbounded. This would allow to replace the smoothness condition of the Shub-Sullivan theorem of [20] by the symmetry to get the same statement as we got for the sphere (Cor. 1.8).

Finally one can ask whether it is reasonable to study maps which are equivariant with respect to actions of other than cyclic groups which act freely on S^n. An explanation is given below.

Remark 7.4. Suppose that f is equivariant with respect to a free action of an arbitrary compact Lie group G. Then for any element $g \in G$ of prime order we may restrict the action to the cyclic group $\{g\}$. Such an element always exists - for finite $\{g\}$ it follows from the Cauchy theorem, for $\{g\}$ infinite it is enough to consider the maximal torus of G. It is obvious that f is $\{g\}$-equivariant, consequently we have a chaos in the sense considered here. On the other hand there are very few finite groups G acting on the sphere freely (e.g. for such G, if $H \subset G$ is an abelian subgroup, then it is cyclic), and there are only three, up to isomorphism, infinite compact Lie groups (S^1, $N(S^1)$ - the normalizer of S^1 in S^3, and S^3) which act freely on the sphere (cf. [3] III 8 for more information). With respect to this, it is more natural to assume that f commutes with a free homeomorphism g of finite order. Moreover if $f : S^n \to S^n$ is G-equivariant with respect to an infinite compact Lie group G, then $\deg(f) = \pm 1$ and the assumption of Theorem 1.6 can not be satisfied.

Remark 7.5. Also in the case of an arbitrary manifold X with a free action of a compact Lie group G it is reasonable to assume that G is finite. Indeed, otherwise for every equivariant map $f : X \to X$ and any k, we have $L(f^k) = 1$.

Consequently, the study of the dynamics of equivariant self-maps by tools which make use of the assumption that $\{L(f^k)\}$ is unbounded is rather ineffectual.

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